REPRESENTATIONS OF PREFERENCE ORDERINGS BY COHERENT UPPER AND LOWER PREVISIONS DEFINED WITH RESPECT TO HAUSDORFF OUTER AND INNER MEASURES

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REPRESENTATIONS OF PREFERENCE ORDERINGS BY COHERENT UPPER AND LOWER PREVISIONS DEFINED WITH RESPECT TO HAUSDORFF OUTER AND INNER MEASURES

-Complex decisions and their integral representation
-Coherent upper conditional previsions defined by Hausdorff outer measures
- Preference Orderings

In a metric space

- **A new model of coherent upper conditional prevision based on Hausdorff outer measures is introduced.**

If the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension

- **The given coherent upper conditional prevision is proven to be monotone, comonotonically additive, submodular and continuous from below.**

If the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension

- **A coherent upper conditional prevision is characterized as the Choquet integral with respect to the upper conditional probability defined by its associated Hausdorff outer measure.**

This presentation consists of three parts:

- Complex decisions and their integral representations
- Coherent upper conditional previsions defined by Hausdorff outer measures
- Preference orederings between random variables

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-Complex decisions and their integral representation
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Complex decisions

Coherent upper conditional previsions

Motivations

The model

Preference orderings

Main results

A complex decision is a decision where the best alternative cannot be obtained by a preference ordering represented by a linear functional

A functional $\mathbf{\Gamma}$ defined on the class L(B) of all random variables defined on a non-empty set B is linear if for every $\boldsymbol{X}, \boldsymbol{Y} \in$ L(B) and for every $\alpha, \beta \in \mathcal{R}$

$$\mathbf{\Gamma}(\alpha\boldsymbol{X} + \beta\boldsymbol{Y}) = \alpha\mathbf{\Gamma}(\boldsymbol{X}) + \beta\mathbf{\Gamma}(\boldsymbol{Y})$$

A preference ordering on the class L(B) of all random variables defined on B is represented by a linear functional $\mathbf{\Gamma}$(e.g. the weighted sum) if and only if

$$\boldsymbol{X} \succ \boldsymbol{Y} \Leftrightarrow \boldsymbol{\Gamma}(\boldsymbol{X}) > \boldsymbol{\Gamma}(\boldsymbol{Y})$$

and

$$\boldsymbol{X} \simeq \boldsymbol{Y} \Leftrightarrow \boldsymbol{\Gamma}(\boldsymbol{X}) = \boldsymbol{\Gamma}(\boldsymbol{Y})$$

Nevertheless not all preference orderings can be represented by a linear functional

Example 1 Let Ω be a non-empty set, **B** = $\{B_1, B_2\}$ and let μ be a probability measure defined on the field generated by **B**. Let consider the class $K = \{X_1, X_2, X_3\}$ of bounded **B**-measurable random variables defined on **B**
by

Random variables	B_1	B_2
X_1	0.3	0.3
X_2	0.7	0
X_3	0	0.7

The preference ordering $X_1 \succ X_2$ and $X_2 \simeq X_3$ cannot be represented by the linear functional (the weighted sum)

$$\Gamma(X) = \sum_{i=1}^{2} \mu(B_i) x_i$$

since there exists no probability measure μ such that the following system has solution:

$$\begin{cases} X_1 \succ X_2 \\ X_1 \simeq X_2 \end{cases} \Leftrightarrow \begin{cases} 0.3\mu(B_1) + 0.3\mu(B_2) & > & 0.7\mu(B_1) + 0\mu(B_2) \\ 0.7\mu(B_1) + 0\mu(B_2) & = & 0\mu(B_1) + 0.7\mu(B_2) \\ \mu(B_1) + \mu(B_2) & = & 1 \end{cases}$$

Remark If only the random variables X_1 and X_2 are considered we have that the preference $X_1 \succ X_2$ can be represented by a linear functional, since

$$X_1 \succ X_2 \Leftrightarrow \begin{cases} 0.3\mu(B_1) + 0.3\mu(B_2) & > & 0.7\mu(B_1) + 0\mu(B_2) \\ \mu(B_1) + \mu(B_2) & = & 1 \end{cases}$$

and the system has solutions: all pair $\left(\mu(B_1); \mu(B_2)\right)$with $\mu(B_1) < \frac{3}{7}$ and $\mu(B_2) = 1 - \mu(B_1)$.

The previous example put in evidence the necessity to introduce non-linear functionals to represent preference orderings and to investigate equivalent random variables.

For B in ***B*** let $X|B$ be the restriction to B of a random variable X defined on Ω and let sup $X|B$ be the supreme value assumed by X on B.

For B in ***B*** and $X|B$ in **K**(B) a coherent upper conditional prevision $\overline{P}(X|B)$ is a real functional on **K**(B) such that the following conditions hold for every $X|B$ and $Y|B$ in **K**(B) and positive constant λ:

1. $\overline{P}(X|B) \leq \sup X|B$

2. $\overline{P}(\lambda X|B) = \lambda \overline{P}(X|B)$ positive homogeneity

3. $\overline{P}(X+Y|B) \leq \overline{P}(X|B) + \overline{P}(Y|B)$ subadditivity

4. $\overline{P}(I_B|B) = 1$

If $\overline{P}(X|B)$ is a coherent upper conditional prevision on a linear space **K**(B) then its conjugate coherent lower conditional prevision is defined by $\underline{P}(X|B) = -\overline{P}(-X|B)$ and

$$\underline{P}(X|B) \leq \overline{P}(X|B)$$

If for every $X|B$ belonging to **K**(B) we have

$$P(X|B) = \underline{P}(X|B) = \overline{P}(X|B)$$

then $P(X|B)$ is called a coherent linear conditional prevision and it is a linear, positive functional on **K**(B).

For each X in K(**B**) let $\overline{P}(X|\boldsymbol{B})$ be the function defined on Ω by

$$\overline{P}(X|\boldsymbol{B})(\omega) = \overline{P}(X|B) \qquad \text{if } \omega \in B.$$

$\overline{P}(X|\boldsymbol{B})$ is called a *coherent upper conditional prevision* and it is coherent if all the $\overline{P}(X|B)$ are coherent.

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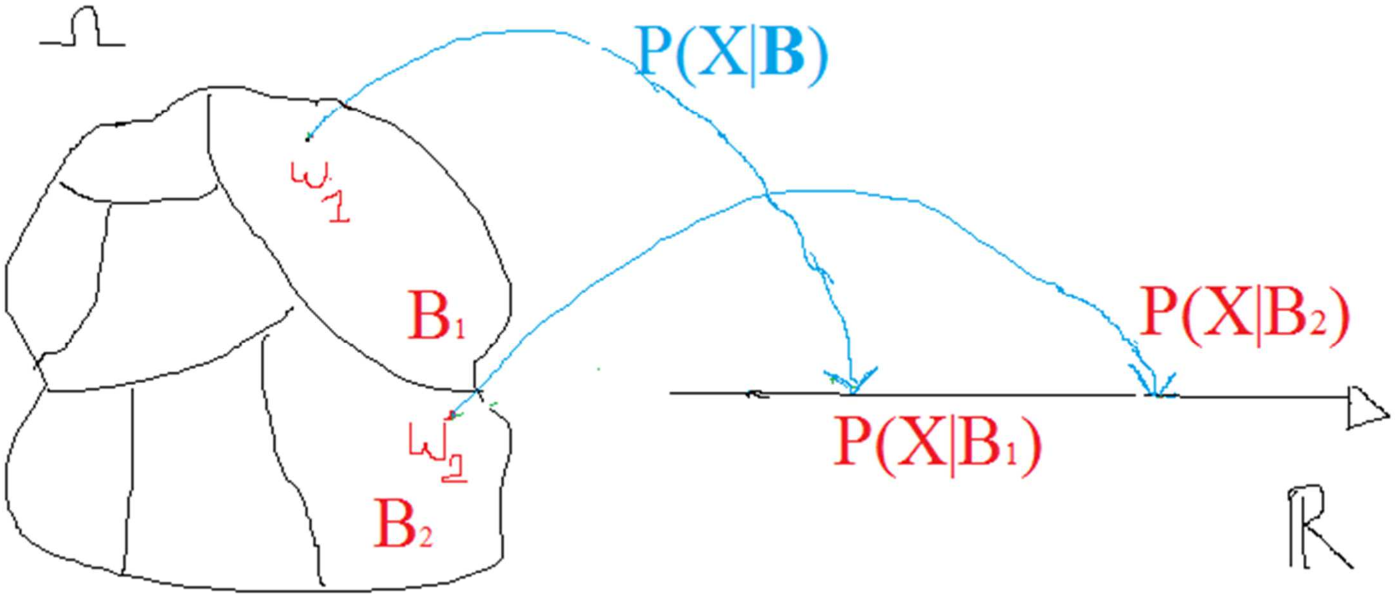


A necessary and sufficient condition for an upper prevision $\overline{P}$ to be coherent is to be the *upper envelope* of linear previsions, i.e. there exists a class M of linear previsions, defined on a same domain, such that

$$\overline{P}(X) = \sup\{P(X) : P \in M\}$$

A new model of coherent upper conditional previsions

Coherent upper conditional prevision defined by its associated Hausdorff outer measure

A random variable X on Ω is **B-**measurable or measurable with respect to the partition **B** if it is constant on the atoms of the partition **B** (Walley 1991, p.291).

Necessary condition for coherence

If for every $B \in \boldsymbol{B}$ $P(X|B)$ are coherent linear conditional previsions and X is ***B***-measurable then (Walley 1991, p. 292)

$$P(X|\boldsymbol{B}) = X$$

Motivations

Why the necessity to propose a new model of coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$?

Because coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ cannot always be defined as extension of conditional expectation $E(X|\boldsymbol{G})$ of measurable random variables.

In fact conditional expectation $E(X|\boldsymbol{G})$ defined by the Radon-Nikodym derivative, according to the axiomatic definition, may fail to be separately coherent.

It occurs because one of the defining properties of the Radon- Nikodym derivative, that is to be measurable with respect to the sigma-field of the conditioning events, contradicts a necessary condition for the coherence ($P(X|\boldsymbol{B}) = X$).

Linear separately coherent conditional prevision $P(X|\boldsymbol{B})$ can be compared with conditional expectation $E(X|\boldsymbol{G})$ if the partition ***B*** generates the sigma-field ***G*** (Koch, 1997, p.262)

$$E(X|\boldsymbol{G})(\omega) = P(X|B) \qquad \text{if } \omega \in B \quad \text{with } B \in \boldsymbol{B}.$$

Definition of conditional expectation(Billingsley, 1986)

Let$\mathbf{F}$ and $\mathbf{G}$ be two sigma-fields of subsets of $\mathbf{\Omega}$ with $\mathbf{G} \subset \mathbf{F}$ and let X be an integrable, $\mathbf{F}$-measurable random variable. Let P be a probability measure on $\mathbf{F}$; define a measure ν on $\mathbf{G}$ by

$$\nu(G) = \int_G XdP.$$

This measure is finite and absolutely continuous with respect to P (i.e. $P(A) = 0 \implies \nu(A) = 0 \quad \forall\, A \in \mathbf{G}$).

Thus there exists a function, the Radon-Nikodym derivative, denoted by $E(X|\boldsymbol{G})$, $\boldsymbol{G}$-measurable, integrable and satisfying the functional equation

$$\int_G E(X|\boldsymbol{G})dP = \int_G XdP \qquad \text{with } G \text{ in } \boldsymbol{G}.$$

This function is unique up to a set of P-measure zero and it is a version of the conditional expectation value.

If linear conditional prevision $P(X|\boldsymbol{B})$ is defined by the Radon-Nikodym derivative the necessary condition for coherence $P(X|\boldsymbol{B}) = X$ is not always satisfied.

Theorem 1 Let $\Omega = [0,1]$, let $\mathbf{F}$ be the Lebesgue sigma-field of [0,1] and let P be the Lebesgue measure on $\mathbf{F}$. Let $\mathbf{G}$ be a sub sigma-field properly contained in $\mathbf{F}$ and containing all singletons of [0,1]. Let ***B*** be the partition of all singletons of [0,1] and let X be the indicator function of an event *A* belonging to $\mathbf{F}$-$\mathbf{G}$. If we define the linear conditional prevision

$P(X|\{\omega\})$ equal to the Radon-Nikodym derivative with probability 1, that is

$$P(X|\{\omega\}) = E(X|\boldsymbol{G})$$

except on a set *N* of [0,1] of P-measure zero, then the conditional prevision $P(X|\{\omega\})$ is not coherent.

(Theorem 1 S. Doria, Annals of Operation Research, Vol. 195, pp.38-44, 2012)

If the equality $P(X|\{\omega\}) = E(X|\boldsymbol{G})$ holds with probability 1, the linear conditional prevision $P(X|\{\omega\})$ is different from X, the indicator function of A. In fact having fixed A in $\mathbf{F}$-$\mathbf{G}$ the indicator function of A is not $\mathbf{G}$-measurable.

It occurs because for every Borel set C

$$X^{-1}(C) = \{\omega \in \Omega : X(\omega) \in C\} =$$

$$= \begin{cases} \emptyset & \text{if } 0, 1 \notin C \\ A & \text{if } 1 \in C \text{ and } 0 \notin C \\ A^c & \text{if } 0 \in C \text{ and } 1 \notin C \\ \Omega & \text{if } 0, 1 \in C \end{cases}$$

and since A $\notin$ **G** then X is not **G**-measurable.

Example 1(Billingsley, 1986; Seidenfeld et al. 2001)

Let Ω = [0,1]

F=Borel sigma-field of Ω,

P = the Lebesgue measure on**F**

G = the sub sigma-field of sets that are either countable or co-countable

B = the partition of all singletons of [0,1].

If the linear conditional prevision is equal, with probability 1, to conditional expectation defined by the Radon-Nikodym derivative, we have that

$$P(X|\boldsymbol{B}) = E(X|\boldsymbol{G}) = P(X)$$

since the events in $\boldsymbol{G}$ have probability either 0 or 1.

So when X is the indicator function of an event A=[a,b] with

0 <a < b < 1 then

$$P(X|\boldsymbol{B}) = P(A)$$

and it does not satisfy the necessary condition for coherence, that is

$$P(X|\{\omega\}) = X.$$

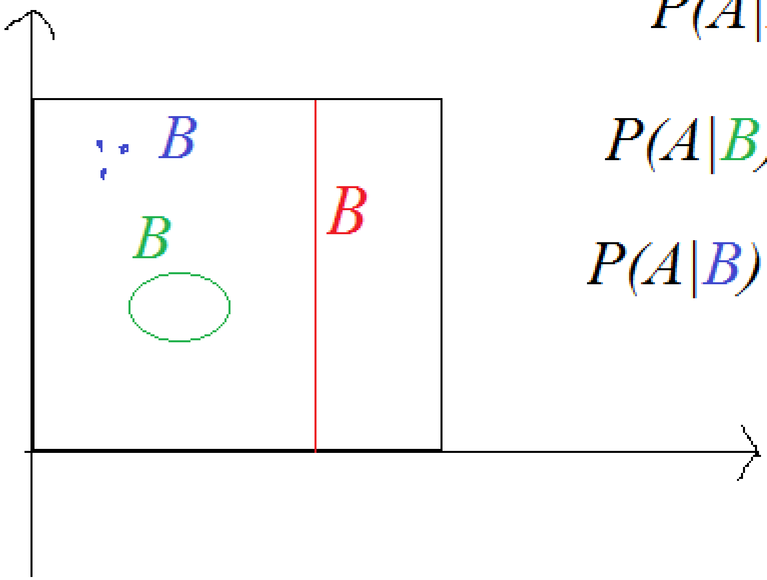
Evident from Theorem 1 and Example 1 is the necessity to introduce a new mathematical tool to define coherent conditional previsions.

The model

Let (Ω, d) be a metric space.

For every $B \in \boldsymbol{B}$ denote by *s* the Hausdorff dimension of the conditioning event *B* and by h^s the Hausdorff s-dimensional outer measure.

h^s is called the Hausdorff outer measure *associated* with the coherent upper conditional prevision $\overline{P}(X|B)$.



$$P(A|B)=h_1(AB)|h_1(B)$$

$$P(A|B)=h_2(AB)|h_2(B)$$

$$P(A|B)=h_0(AB)|h_0(B)$$

Theorem 2. Let **L**(B) be the class of all bounded random variables on B and let m be a 0-1-valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B. Then for each $B \in \boldsymbol{B}$ the functional $\overline{P}(X|B)$ defined on **L**(B) by

$$\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s \qquad \text{if } 0 < h^s(B) < +\infty$$

$$\overline{P}(X|B) = m(XB) \qquad \text{if } \; h^s(B) = 0, +\infty$$

is a coherent upper conditional prevision.

(Theorem 2, S. Doria, Annals of Operation Research, Vol. 195, pp.38-44, 2012)

The unconditional coherent upper prevision $\overline{P}(X|\Omega)$ is obtained as a particular case when the conditioning event is Ω.

Coherent upper conditional probabilities $\overline{P}(A|B)$ are obtained when only 0-1 valued random variables are considered.

Theorem 3 Let m be a 0-1-valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B. Then for each $B \in \boldsymbol{B}$ the function $\overline{P}(A|B)$ defined on $\wp(B)$ by

$$\overline{P}(A|B) = \frac{h^s(AB)}{h^s(B)} \qquad if \quad 0 < h^s(B) < +\infty$$

and by

$$\overline{P}(A|B) = m(AB) \qquad if \quad h^s(B) = 0, +\infty$$

is a coherent upper conditional probability.

Main Results

Let B be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension.

Denote by

$$\mu_B^*(A) = \overline{P}(A|B) = \frac{h^s(AB)}{h^s(B)}$$

the coherent upper conditional probability defined on $\wp(B)$.

From Theorem 2 we have that the coherent upper conditional prevision $\overline{P}(\cdot\,|B)$ is a functional on $\mathbf{L}(B)$ with values in $\mathbb{R}$ and the coherent upper conditional probability μ_B^* integral represents $\overline{P}(X|B)$ since

$$\overline{P}(X|B) = \int X d\mu_B^* = \frac{1}{h^s(B)} \int_B X dh^s$$

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If the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension and **K**(B) is a linear lattice of bounded random variables containing all constants

Necessary and sufficient conditions for a coherent upper conditional prevision to be uniquely represented as the Choquet integral with respest to the upper conditional probability defined by its associated Hausdorff outer measure are to be :

Monotone

Comonotonically additive

Submodular

Continuous from below

Uniqueness of the representing set function

(Denneberg, 1994, Proposition 13.5)

If a functional Γ, defined on a domain L is monotone, comonotonically additive, submodular and continuous from below then Γ is representable as Choquet integral with respect to a monotone, submodular set function, which is continuous from below.

Furthermore all monotone set functions on $\wp(\Omega)$ with these properties agree on the set system of weak upper level set

$$M = \{\{\omega \in \Omega : X(\omega) \geq x\} : X \in L; x \in \mathbb{R}^+\}$$

Example 1 (continued)

We consider the following class of probability measures

	B_1	B_2
P_1	**1**	**0**
P_2	**0**	**1**
$\overline{\mu}$	**1**	**1**
$\underline{\mu}$	**0**	**0**

Let $\overline{\mu}_B$ be the coherent upper conditional probability defined by

$$\overline{\mu} \quad (A) = max\{P_1(A), P_2(A)\}$$

and let $\underline{\mu}_B$ be the coherent lower conditional probability defined by

$$\underline{\mu} \quad (A) = min\{P_1(A), P_2(A)\}$$

The coherent upper and lower conditional previsions can be represented as Choquet integral.

If the atoms B_i are enumerated so that $x_i = X(B_i)$ are in in descending order, i.e.

$x_1 \geq x_2 \geq \cdots \geq x_n$ and $x_{n+1} = 0$ the Choquet integral with respect to μ is given by

$$C\int X d\mu = \sum_{i=1}^{n} (x_i - x_{i+1})\mu(S_i)$$

where $S_i = B_1 \cup B_2 \cup \ldots \cup B_i$.

In Example 1 we have

$$\underline{P}(X_1) = 0.3 \quad \text{and} \quad \underline{P}(X_2) = \underline{P}(X_3) = 0$$

so that the preference ordering

$$X_1 \succ X_2 \text{ and } X_2 \simeq X_3.$$

can be represented by the coherent lower prevision $\underline{P}$.

The preference ordering cannot be represented by the coherent upper prevision $\overline{P}$ since

$$\overline{P}(X_1) = 0.3 \quad \text{and} \quad \overline{P}(X_2) = \overline{P}(X_3) = 0.7$$

The preference ordering $X_1 \succ X_2$ and $X_2 \simeq X_3$
can be also represented by the vacuous lower probability defined by

$$\underline{P}(X|\Omega) = inf\{X(\omega) : \omega \in \Omega\}$$

In fact

$$\underline{P}(X_1|\Omega) = 0.3 \quad \text{and} \quad \underline{P}(X_2|\Omega) = \underline{P}(X_3|\Omega) = 0$$

and also in this case $\underline{P}(I_{B_1}) = \underline{P}(I_{B_2}) = 0.$

The previous coherent lower previsions which represent the ordering

$$X_1 \succ X_2 \text{ and } X_2 \simeq X_3$$

do not satisfy the following disintegration property for every X in K

$$\underline{\boldsymbol{P}}(X|\Omega) = \sum_{B \in \boldsymbol{B}} \underline{\boldsymbol{P}}(B)\underline{\boldsymbol{P}}(X|B).$$

In multi-criteria decision problem denoted by Ω the set of criteria, the elements of a partition **B** can represent clusters or macro-criteria- which are representative of the general objectives of the decision problem, as goals to pursue through the implementation of specific policies - and the elements in each B ∈ $\boldsymbol{B}$ are the criteria.
To determine the best alternative with respect to all criteria we should require that the non-linear functional which

represent the preference ordering, satisfies the disintegration property so that we can compare all the alternatives on each cluster and then to aggregate the obtained results.

Orderings represented by coherent lower and upper conditional previsions

A *strict ordering* (i.e. *antisymmetric* and *transitive* binary relation) induced by a coherent lower conditional prevision $\underline{P}(\cdot|B)$ can be defined on the class of random variables belonging to L(B) (Walley (Section 3.8.1)):

Definition 1 We say that the random variable X_i is *preferable* to X_j given B with respect to $\underline{P}(\cdot\,|B)$, i.e. $X_i \succ_* X_j$ given B if and only if

$$\underline{P}\left(X_i - X_j \middle| B\right) > 0.$$

Some information can be lost when a strict preference order is defined by to $\underline{P}(\cdot\,|B)$ since $\underline{P}$ does not contain any information about which random variable, with $\underline{P}(X|B) = 0$ are really desirable.

In Walley (section 3.9.4) the following definitions are given.

Definition 2 Let K be a finite class of random variables. A random variable X_i in K is *inadmissible* in K given B if there is X_j in K such that$X_j \succ_* X_i$ with j≠i. Otherwise X_i is *admissibile* in K.

X_i is admissible in K⇔ $\underline{P}\left(X_j - X_i\right) < 0 \forall X_j \in K \; with \; i \neq j$.

A *weak ordering* (i.e. *riflexive* and *transitive* binary relation) induced by a coherent upper conditional prevision $\overline{P}(\cdot\ |B)$ can be defined on the class of random variables belonging to L(B) (Doria, 2014):

Definition 2 We say that the random variable X_i *is preferable* to X_j given B with respect to $\overline{P}(\cdot\,|B)$, i.e. $X_i \succ_* X_j$ given B if and only if

$$\overline{P}(X_i - X_j|B) > 0.$$

Definition 3 We say that the random variable X_i and X_j are *equivalent* given B with respect to $\overline{P}(\cdot\,|B)$, i.e. $X_i \approx * X_j$ given B if and only if

$$\overline{P}(X_i|B) = \overline{P}(X_j|B).$$

Definition 4 We say that the random variable X_i and X_j are *indifferent* given B with respect to $\overline{P}(\cdot\,|B)$, i.e. $X_i \approx X_j$ given B if and only if

$$\overline{P}\left(\left(X_i - X_j\right)|B\right)\right) = \overline{P}\left(\left(X_j - X_i\right)|B\right)\right) = 0.$$

Definition 5 An admissible random variable X_i in K is *maximal* in K given B under the coherent lower prevision $\underline{P}(\cdot\,|B)$ when X_i is admissible in K and

$$\overline{P}\left(X_i - X_j \middle| B\right) \geq 0 \;\forall X_j \text{ in K.}$$

If $P(\cdot\,|B)$ is a linear prevision, the maximal random variable belonging to L(B) with respect to $P(\cdot\,|B)$ is the admissible random variable X_i which satisfies

$$P(X_i|B) \geq P\left(X_j\middle|B\right) \text{ for all } X_j \text{ in K.}$$

Any alternative which maximizes $P(X_j|B)$ over X_j in K is called a *Bayes random variable* under $P(\cdot\,|B)$.

A Bayes random variable under a coherent lower conditional prevision is a random variable which is maximal under a linear prevision on the class of all random variables defined on B.

Definition 3 An admissible random variable X_i is defined to be *a Bayes random variable* under a coherent lower prevision $\underline{P}$ when, for each X_j in K there is a linear prevision $P \in M(\underline{P})$ such that $P(X_i|B) \geq P(X_j|B)$ $\forall X_j$ in K.

If X_i is maximal under some $P \in M(\underline{P})$ then

$P(X_i|B) \geq P(X_j|B)\ \forall X_j$ in K so

$$\overline{P}(X_i - X_j|B) \geq P(X_i - X_j|B) = P(X_i) - P(X_j) \geq 0$$

and X_i is maximal under $\underline{P}$.

So a Bayes random variable under a coherent lower prevision $\underline{P}$ is maximal under $\underline{P}$ but the converse is not true.

$$X \text{ is a Bayes random variable under } \underline{P} \Rightarrow X \text{ is maximal under } \underline{P}$$

$$\not\Leftarrow$$

Example 1 (continued)

Random variables	B_1	B_2
X_1	0.3	0.3
X_2	0.7	0
X_3	0	0.7

We consider the following class of probability measures

	B_1	B_2
P_1	**1**	**0**
P_2	**0**	**1**
$\overline{\mu}$	**1**	**1**
$\underline{\mu}$	**0**	**0**

We obtain

$$\underline{P}(X_i - X_j) = C\int (X_i - X_j)d\underline{\mu} < 0 \quad \text{for all } i,j \in \{1,2,3\} \text{ with } i \neq j$$

so all random variables X_i for all $i \in \{1,2,3\}$ are admissible

and

$$\overline{P}(X_i - X_j) = C\int (X_i - X_j)d\overline{\mu} \geq 0 \quad \text{for all } i,j \in \{1,2,3\}$$

so all random variables X_i for all $i \in \{1,2,3\}$ are maximal.

No random variable X_i for all $i \in \{1,2,3\}$ is a Bayes random variable.

According to the model based on Hausdorff outer measures,

by Theorem 1 , if all the conditioning events has Hausdorff measure in its Hausdorff dimension equal to 0 or $+\infty$,a coherent conditional prevision can be defined on a 0-1 valued measure *m* finitely additively but not countably additive on P(Ω).

Example

Let (Ω, d) be a metric space with $\Omega = N$ so that $dim_H \Omega$=0 and $h^0(\Omega) = +\infty$. Let $\mathbf{B} = \{B_1, B_2\}$ be the partition of Ω where $B_1 = \{p \in N: p = 2n; n \in N\}$ and $B_2 = \{d \in N: d = 2n - 1; n \in N\}$ so that $dim_H \mathrm{B}_1$=$dim_H \mathrm{B}_2$=0

and $h^0(\mathrm{B}_1) = h^0(\mathrm{B}_1) = +\infty$

We consider the following probability measures

	$h^s(B_1) = +\infty$	$h^s(B_2) = +\infty$
$P(X\|B_1) = m_{B_1}$		
$P(X\|B_2) = m_{B_2}$		

We can chose m_{B_1} such that

$$P(X_1|B_1) = 1;\; P(X_2|B_1) = 0;\; P(X_3|B_1\,) = 0$$

We can choose m_{B_2} such that

$$P(X_1|B_2) = 1;\ \ P(X_2|B_2) = 0;\ P(X_3|B_2\ ) = 0$$

we can choose m_Ω such that

$$P(X_1|\Omega) = 1;\ \ P(X_2|\Omega) = 0;\ P(X_3|\Omega\ ) = 0$$

and $P(B_1|\Omega)$=0 and $P(B_2|\Omega) = 1$ so that

the disintegration property holds since

$$P(X_1|\Omega) = P(B_1|\Omega)P(X_1|B_1) + P(B_2|\Omega)P(X_1|B_2)$$

$$1 = 0 \cdot 1 + 1 \cdot 1$$

$$P(X_2|\Omega) = P(B_1|\Omega)P(X_2|B_1) + P(B_2|\Omega)P(X_2|B_2)$$

$$0 = 0 \cdot 1 + 1 \cdot 0$$

$$P(X_3|\Omega) = P(B_1|\Omega)P(X_3|B_1) + P(B_3|\Omega)P(X_3|B_2)$$

$$0 = 0 \cdot 1 + 1 \cdot 0.$$

The ordering

$$X_1 \succ X_2 \text{ and } X_2 \simeq X_3$$

can be represented by the coherent conditional prevision (defined in Theorem 1)
and it holds with respect to Ω and with respect to B_1 and to B_2.
X_1 is a maximal and a Bayes random variable with respect to m_Ω.

Maximal random variables and Bayes random variables

Theorem 1

Let $K \subset L(B)$ be a class *of comonotonic random variables* and let μ be a *submodular* coherent upper conditional probability defined on $\wp(B)$ and let $\overline{P}(\cdot\,|B)$ be a coherent upper conditional prevision defined as Choquet integral with respect to μ then a random variable $X \in K$ is maximal in *K* if and only if X is a Bayes random variable in *K*.

Theorem 2

Let $K \subset L(B)$ be a class *of* random variables and let $X_i \in K$ such that the class $C = \{X_i - X_j;\ X_j \in K\}$ is comonotonic. Let μ be a *submodular* coherent upper conditional probability defined on $\wp(B)$ and let $\overline{P}(\cdot\,|B)$ be a coherent upper conditional prevision defined as Choquet integral with respect to μ then a random variable $X \in K$ is maximal in *K* with respect to the conjugate lower prevision $\underline{P}(\cdot\,|B)$ if and only if X is a Bayes random variable in *K*.

Theorem 3

Let $K = \{X_i, X_j\} \subset L(B)$ be a class *containing only two* random variables. Let μ be a *submodular* coherent upper conditional probability defined on $\wp(B)$ and let $\overline{P}(\cdot|B)$ be a coherent upper conditional prevision defined as Choquet integral with respect to μ then a random variable $X \in K$ is maximal in *K* with respect to the conjugate lower prevision $\underline{P}(\cdot|B)$ if and only if X is a Bayes random variable in *K*.

Thank you for your attention!

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